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Alternating Polynomials Associated with the Chebyshev Extrema Nodes

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By analogy with Lagrange interpolation, the fundamental alternating polynomials are introduced. The interlacing property of the zeros of these polynomials corresponding to the Chebyshev extrema nodes is established. The behavior of the corresponding Lebesgue function is studied. In this study our main tool is a relationship between this function and the corresponding Lebesgue function induced by the interpolation. Taking advantage of our previous result concerning interpolation, a new estimate for the norm of the alternating operator is obtained. This result gives an affirmative answer to a question posed by Cheney and Rivlin.

1. INTRODUCTION

Let $X=\{x_k\}_{k=0}^{n+1}$, $-1\leqslant x_{n+1}<x_n\cdots<x_1<x_0\leqslant 1$ be a set of $n+2$ distinct points in $[-1, 1]$, and denote by $C[-1, 1]$ the Banach space of continuous functions on $[-1, 1]$ equipped with the uniform norm. To each $f(x)\in C[-1, 1]$ there corresponds a unique interpolation polynomial

$$L_{n+1}(X; x)=\sum_{k=0}^{n+1} f(x_k)\, l_k(X; x), \tag{1}$$

where

$$l_k(X; x)=\prod_{\substack{i=0\\ i\neq k}}^{n+1} (x-x_i)/(x_k-x_i).$$

$L_{n+1}(X)$ may be interpreted as a projection of $C[-1, 1]$ onto the subspace π_{n+1} consisting of all polynomials of degree $\leqslant n+1$. In [3] the author introduced the (generalized) alternating polynomials $A_n(X; x)$ which are

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related to the interpolation polynomials induced by the same set of nodes in the following way:

$$A_n(X; x) = L_{n+1}(X; x) - \frac{f[x_0, x_1, ..., x_{n+1}]}{2^n} T_{n+1}(x), \tag{2}$$

where $f[x_0, x_1, ..., x_{n+1}]$ is the divided difference of $f(x)$ on the point set X and $T_n(x) = \cos(n \arccos x)$. Relation (2) shows that the A_n-polynomials may be derived from the corresponding interpolating polynomials by applying one step of the Lanczos economization technique. $A_n(X)$ may also be viewed as a projection of $C[-1, 1]$ onto π_n.

Of special importance in applications are sets of nodes satisfying

$$\operatorname{sgn}[T_{n+1}(x_k)] = (-1)^k, \qquad k = 0, 1, ..., n+1. \tag{3}$$

For such sets the A_n-polynomials coincide with the particular case of the next-to-interpolatory polynomials in the sense of Motzkin and Sharma [8] (see [3] for details).

Let A_n-polynomial be represented in the "Lagrangian" form:

$$A_n(X; x) = \sum_{k=0}^{n+1} f(x_k)\, a_k(X; x). \tag{4}$$

Then by analogy with the interpolation, it is natural to call $a_k(X; x)$ the fundamental alternating polynomials. Relation (2) yields

$$a_k(X; x) = l_k(X; x) - \frac{1}{2^n \omega'(x_k)} T_{n+1}(x), \tag{5}$$

where $\omega(x) = \prod_{j=0}^{n+1} (x - x_k)$. The arrangement of the roots of the fundamental polynomials $a_k(X; x)$ is of great importance in our study. It can be easily seen that if, in addition to (3), the nodes $\{x_k\}_{k=0}^{n+1}$ satisfy $|\omega'(x_k)| > 2^{-n}$, $k = 0, 1, ..., n+1$, then each $a_k(X; x)$, $k = 1, 2, ..., n$, has $n-1$ distinct real roots on (x_{n+1}, x_0) and hence an additional root outside $[x_{n+1}, x_0]$. Moreover, all the roots of $a_0(X; x)$ and $a_{n+1}(X; x)$ lie in the interval (x_{n+1}, x_0). In order to obtain additional information concerning the location of the roots of $a_k(X; x)$, we have to restrict ourselves to specific sets of nodes.

In the present paper we deal with the Chebyshev extrema nodes $X = U = \{\cos[k\pi/(n+1)]\}_{k=0}^{n+1}$. In Section 2, two lemmas concerning the fundamental polynomials $a_k(U; x)$ are proved. In Lemma 1, the interlacing property of the roots of the fundamental polynomials $a_k(U; x)$, $k = 0, 1, ..., n+1$, is established, while in Lemma 2 we estimate the "degree of the orthogonality" of $\{a_k(U; x)\}_{k=0}^{n+1}$ with respect to the Chebyshev weight. Section 3 is devoted to the study of the corresponding Lebesgue function defined by $\mu_n(U; x) = \sum_{k=0}^{n+1} |a_k(U; x)|$. Our main tool here is a relationship between $\mu_n(U; x)$ and the corresponding Lebesgue function induced by the interpolation $\lambda_{n+1}(U; x) = \sum_{k=0}^{n+1} |l_k(U; x)|$. By making use of the interlacing property of the roots of $a_k(U; x)$, $k = 0, 1, ..., n+1$, and taking advantage of our previous result concerning interpolation, we obtain a new estimate for the operator norm of $A_n(U)$. This estimate gives an affirmative answer to a question posed by Cheney and Rivlin in [5]. The paper is concluded with an observation concerning the mean square convergence of the $A_n(U; x)$-polynomials. This observation serves as an illustration of the general principle, which says that the laws of the asymptotic distribution of nodes are not fine enough to characterize completely the behavior of the alternating process.

2. The Fundamental Polynomials $a_k(U; x)$

Let $U = \{\eta_k = \cos[k\pi/(n+1)]\}_{k=0}^{n+1}$. It follows from (5) that

$$a_k(U; x) = l_k(U; x) - \gamma_k \frac{(-1)^k}{n+1} T_{n+1}(x) \tag{6}$$

with $\gamma_k = 1$ for $k = 1, 2, ..., n$ and $\gamma_0 = \gamma_{n+1} = \frac{1}{2}$.

The fundamental polynomials $a_k(U; x)$ may also be expressed in terms of the Chebyshev polynomials

$$a_k(U; x) = \frac{2\gamma_k}{n+1} \sum_{m=0}^{n}{}' T_m(\eta_k)\, T_m(x), \tag{7}$$

where $\sum'$ denotes a sum whose first term is halved. By applying to (7) the well-known Christoffel–Darboux identity, one can easily derive the following useful representation which is due to Eterman [6] (see also Meinardus [7]):

$$a_k(U; x) = \frac{\gamma_k(-1)^k [T_n(x) - \eta_k T_{n+1}(x)]}{(n+1)(\eta_k - x)}. \tag{8}$$

We proceed now to prove the following property.

Lemma 1. *Let* $I_j = [\eta_j, \eta_{j-1}]$, $j = 1, 2, ..., n+1$. *Each* $a_k(U; x)$, $k = 0, 1, ..., j-2, j+1, ..., n+1$, *has a real root* $r_k^{(j)}$ *on* I_j. *These roots are ordered as follows*:

$$\eta_j < r_{j+1}^{(j)} < r_{j+2}^{(j)} < \cdots < r_{n+1}^{(j)} < \xi_j < r_0^{(j)} < r_1^{(j)} < \cdots < r_{j-2}^{(j)} < \eta_{j-1}, \tag{9}$$

where $\xi_j = \cos[(2j-1)\pi/(2n+2)]$.

Proof. We start by verifying that the roots of $a_k(U; x)$ for $k \leqslant j-2$ and $k \geqslant j+1$ are separated by the "middle" point ξ_j. Using (8), we get

$$\begin{aligned}
\operatorname{sgn}[a_k(U; \xi_j)] &= (-1)^k \operatorname{sgn}[T_n(\xi_j)] \operatorname{sgn}[\eta_k - \xi_j] \\
&= (-1)^{k+j+1} \operatorname{sgn}[\eta_k - \xi_j] \\
&= (-1)^{k+j+1}, \qquad k = 0, 1, ..., j-2, \\
&= (-1)^{k+j}, \qquad k = j+1, ..., n+1.
\end{aligned} \tag{10}$$

Next, by virtue of (6), $\operatorname{sgn}[a_k(U; \eta_j)] = (-1)^{k+j+1}$, $j \neq k$. and hence $r_k^{(j)} \in (\eta_j, \xi_j)$ for $k = j+1, ..., n+1$, while $r_k^{(j)} \in (\xi_j, \eta_{n-1})$ for $k = 0, 1, ..., j-2$. Thus to prove the lemma it suffices to show that $\operatorname{sgn}[a_k(U; r_{k+1}^{(j)})] = (-1)^{k+j}$, $k = 0, 1, ..., j-3, j+1, ..., n$. Representation (8) reveals

$$\begin{aligned}
\operatorname{sgn}[a_k(U; r_{k+1}^{(j)})] &= (-1)^k \operatorname{sgn}[T_n(r_{k+1}^{(j)}) - \eta_k T_{n+1}(r_{k+1}^{(j)})] \operatorname{sgn}[\eta_k - r_{k+1}^{(j)}] \\
&= (-1)^{k+1} \operatorname{sgn}[T_{n+1}(r_{k+1}^{(j)})] \operatorname{sgn}[\eta_k - r_{k+1}^{(j)}].
\end{aligned} \tag{11}$$

We have used in (11) the fact that $r_{k+1}^{(j)}$ is the root of $a_{k+1}(U; x)$ and hence $T_n(r_{k+1}^{(j)}) - \eta_{k+1} T_{n+1}(r_{k+1}^{(j)}) = 0$. It remains to note that for $k = 0, 1, ..., j-2$, $(-1)^{j-1} T_{n+1}(r_{k+1}^{(j)}) > 0$ and $\eta_k - r_{k+1}^{(j)} > 0$, while for $k = j+1, ..., n+1$, $(-1)^j T_{n+1}(r_{k+1}^{(j)}) > 0$ and $\eta_k - r_{k+1}^{(j)} < 0$. This concludes the proof of Lemma 1.

In the next lemma, the "degree of the orthogonality" of the fundamental polynomials $a_k(U; x)$, $k = 0, 1, ..., n+1$, with respect to the Chebyshev weight is estimated.

LEMMA 2.

$$\int_{-1}^{1} (1-x^2)^{-1/2} a_k(U; x)\, a_l(U; x)\, dx = \begin{cases} \dfrac{(-1)^{k+l+1} \gamma_k \gamma_l \pi}{(n+1)^2}, & k \neq l, \\[2ex] \dfrac{\gamma_k \pi}{n+1} \left(1 - \dfrac{\gamma_k}{n+1} \left(1 - \dfrac{\gamma_k}{n+1}\right)\right), & k = l. \end{cases} \tag{12}$$

Proof. Upon using representation (7) and well-known properties of the Chebyshev polynomials $T_n(x)$, we obtain

$$\begin{aligned}
\int_{-1}^{1} (1-x^2)^{-1/2} a_k(U; x)\, a_l(U; x)\, dx &= \frac{2\gamma_k \gamma_l \pi}{(n+1)^2} \sum_{m=0}^{n}{}' T_m(\eta_k)\, T_m(\eta_l) \\
&= \frac{\gamma_l \pi}{n+1} a_k(U; \eta_l).
\end{aligned} \tag{13}$$

On the other hand, relation (6) reveals

$$a_k(U;\eta_l)=\begin{cases} =\dfrac{(-1)^{k+l+1}\gamma_k}{n+1}, & l\neq k,\\ 1-\dfrac{\gamma_k}{n+1}, & l=k. \end{cases} \tag{14}$$

Combining (13) and (14) completes the proof of the lemma.

3. The Lebesgue Function $\mu_n(U;x)$

By analogy with the interpolation, the Lebesgue function $\mu_n(X;x)$ associated with the alternating operator $A_n(X)$ is defined as

$$\mu_n(X;x)=\sum_{k=0}^{n+1}|a_k(X;x)|. \tag{15}$$

It is known (see, e.g., [4]) that the operator norm of $A_n(X;\cdot)$, as an operator on $C[-1,1]$, equals the sup norm of its Lebesgue function:

$$\|A_n(X)\|=\max_{-1\leqslant x\leqslant 1}\mu_n(X;x). \tag{16}$$

In this section we study the behavior of $\mu_n(U;x)$. Our main tool is a relationship between this function and the corresponding Lebesgue function $\lambda_{n+1}(U;x)$ induced by the interpolation. As before, we denote by $r_k^{(j)}$ the root of $a_k(U;x)$ lying on $I_j=[\eta_j,\eta_{j-1}]$, $j=1,2,...,n+1$. Our attention will be restricted to the study of $\mu_n(U;x)$ on the "middle" subintervals $I_{j,0}=[r_{n+1}^{(j)},r_0^{(j)}]$. By virtue of Lemma 1,

$$\mu_n(U;x)=\sum_{\nu=0}^{j-1}(-1)^{j+\nu+1}a_\nu(U;x)+\sum_{\nu=j}^{n+1}(-1)^{j+\nu}a_\nu(U;x),\qquad x\in I_{j,0}. \tag{17}$$

On the other hand, the corresponding Lebesgue function induced by the interpolation $\lambda_{n+1}(U;x)$ may be written as

$$\lambda_{n+1}(U;x)=\sum_{\nu=0}^{j-1}(-1)^{j+\nu+1}l_\nu(U;x)+\sum_{\nu=j}^{n+1}(-1)^{j+\nu}l_\nu(U;x),\qquad x\in I_j. \tag{18}$$

Combining (17), (18), and (6) we obtain, after some simplification,

$$\lambda_{n+1}(U; x) - \mu_n(U; x) = \frac{(-1)^j(n+2-2j)}{(n+1)} T_{n+1}(x), \qquad x \in I_{j,0}. \quad (19)$$

It follows from (19) that $\lambda_{n+1}(U; \xi_j) = \mu_n(U; \xi_j)$, $j = 1, 2, ..., n+1$. (Moreover, if the number of nodes is even, $n = 2N$, then $\lambda_{2N+1}(U; x)$ coincides identically with $\mu_{2N}(U; x)$ for $x \in I_{N,0}$.) Since, on the other hand, $\|L_{2N+1}(U)\| = \lambda_{2N+1}(U; 0)$ (see, e.g., [1]), $\|A_{2N}(U)\| \geqslant \|L_{2N+1}(U)\|$, $N = 0, 1, 2, ...$. This inequality has been noticed by Cheney and Rivlin in [5]. They also remark: "We don't know whether it is also true for the odd case." The following theorem contains an affirmative answer to this question.

THEOREM 1.

$$\|A_n(U)\| \geqslant \|L_{n+1}(U)\|, \qquad n = 0, 1, 2, ... \quad (20)$$

Proof. We need consider only the odd case, $n + 2 = 2N + 1$, $N = 1, 2, ...$. It is known (see, e.g., [1]) that $\lambda_{2N}(U; x)$ attains its maximal value in the interval $I_N = [\eta_N, \eta_{N-1}]$. Therefore, we can restrict ourselves to a comparison of $\mu_{2N-1}(U; x)$ and $\lambda_{2N}(U; x)$ for $x \in I_N$. The idea of the proof is to show that for any $x \in I_N$ there exists a point $z \in I_N$ such that $\mu_{2N-1}(U; z) \geqslant \lambda_{2N}(U; x)$. Denote by $F_{2N-1}(x)$ the continuation of $\mu_{2N-1}(U; x)$, $x \in I_{N,0}$ as a polynomial on the interval I_N. It is clear that $\mu_{2N-1}(U; x) \geqslant F_{2N-1}(x)$, $x \in I_N$, and hence it suffices to show for any $x \in I_N$ the existence of $z \in I_N$ (depending on x) such that $F_{2N-1}(z) \geqslant \lambda_{2N}(U; x)$, $x \in I_N$. To this end we apply the trigonometric substitution $x = \cos(\Theta_{N-1} + \Theta)$, $\Theta_{N-1} = (N-1)\pi/2N$, $0 \leqslant \Theta \leqslant \pi/2N$, and put $\Phi_{2N-1}(\Theta) = F_{2N-1}(\cos[\Theta_{N-1} + \Theta])$, $\lambda_{2N}(U; I_N, \Theta) = \lambda_{2N}(U; \cos[\Theta_{N-1} + \Theta])$. By virtue of (19) we have

$$\Phi_{2N-1}(\Theta) - \lambda_{2N}(U; I_N, \Theta) = \frac{(-1)^{N+1}\cos(2N\Theta)}{2N}, \qquad 0 \leqslant \Theta \leqslant \frac{\pi}{2N}. \quad (21)$$

Now we apply the following formula, which estimates the "degree of asymmetry" of the Lebesgue function $\lambda_{2N}(U; I_n; \Theta)$ (see [2]):

$$\lambda_{2N}\left(U; I_N, \frac{\pi}{2N} - \Theta\right) - \lambda_{2N}(U; I_N, \Theta) = \frac{\sin(2N\Theta)}{2N} \left\{\tan\left(\frac{\pi}{4N} - \frac{\Theta}{2}\right) - \tan\frac{\Theta}{2}\right\}. \quad (22)$$

Upon comparing (22) and (21), we find that the following inequality has to be verified:

$$\frac{\cos(2N\Theta)}{2N}-\frac{\sin(2N\Theta)}{2N}\left\{\tan\left(\frac{\pi}{4N}-\frac{\Theta}{2}\right)-\tan\frac{\Theta}{2}\right\}>0, \qquad 0<\Theta<\frac{\pi}{4N},$$

or equivalently

$$H_{2N}(\Theta)\equiv\cot(2N\Theta)-\tan\left(\frac{\pi}{4N}-\frac{\Theta}{2}\right)+\tan\frac{\Theta}{2}>0, \quad 0<\Theta<\frac{\pi}{2N}. \quad (23)$$

Since $H_{2N}(0)>0$ and $H_{2N}(\pi/4N)=0$, it is sufficient to show that

$$H'_{2N}(\Theta)=\frac{1}{2\cos 2\left(\frac{\Theta}{2}\right)}+\frac{1}{2\cos^2\left(\frac{\pi}{4N}-\frac{\Theta}{2}\right)}-\frac{2N}{\sin^2(2N\Theta)}<0, \quad \Theta\in\left(0,\frac{\pi}{4N}\right).$$

But $\cos^2(\Theta/2)>\cos^2(\pi/4N-\Theta/2)$, $0<\Theta<\pi/4N$, and hence it remains to check that

$$\sin^2(2N\Theta)<2N\cos^2(\pi/4N-\Theta/2), \qquad 0<\Theta<\pi/4N,$$

or, finally, that

$$\sin(2N\Theta)<\sqrt{2N}\cos(\pi/4N-\Theta/2), \qquad \Theta\in(0,\pi/4N), \quad N=1,2,\ldots.$$

This last inequality follows immediately from the fact that

$$\sqrt{2N}\cos(\pi/4N-\Theta/2)>1, \qquad \Theta\in(0,\pi/4N), \quad N=1,2,\ldots.$$

This completes the proof of the theorem.

Remark. For the upper bound of $\|A_n(U)\|$, the following estimate is due to Cheney and Rivlin [5] (see also Phillips and Taylor [10]):

$$\|A_n(U)\|<\|L_{n+1}(U)\|+1. \qquad (24)$$

Notice that this estimate follows immediately from (6). We performed some intensive numerical calculations which indicate that when the number of alternation points is even, $\max_{-1\leqslant x\leqslant 1}\mu_n(U;x)=\mu_n(U;0)$, while for odd number of points, $n+2=2N+1$, the Lebesgue function $\mu_{2N-1}(U;x)$ attains its maximal value in the subinterval $I_{N,0}$. Assuming these facts to be true, one can conclude

$$\|A_{2N}(U)\|=\|L_{2N+1}(U)\|, \qquad N=0,1,\ldots \qquad (25)$$

$$0<\|A_{2N-1}(U)\|-\|L_{2N}(U)\|<\frac{1}{2N}, \qquad N=1,2,\ldots \qquad (26)$$

4. Concluding Remarks

As was mentioned in the Introduction, the $A_n(U; x)$-polynomials may be considered as a special case of the next-to-interpolatory polynomials, which have been introduced and studied by Motzkin and Sharma in [8, 9]. In particular, since the nodes $\eta_k = \cos[k\pi/(n+1)]$, $k = 0, 1, ..., n+1$, are known to coincide with the roots of $T_{n+2}(x) - T_n(x)$, one can apply Theorem 7 of [9] to obtain

$$\lim_{n\to\infty} \int_{-1}^{1} [f(x) - A_n(U; x)]^2 \, dx = 0 \tag{27}$$

for any $f(x) \in C[-1, 1]$. In the above setting, U denotes the infinite matrix of nodes whose nth row is $(\eta_{n+1}, \eta_n, ..., \eta_0)$. Note also that (27) may be proved directly by applying to $A_n(U; x)$ the standard Fejer technique and making use of Lemma 2. On the other hand, it was shown in [9] that for the matrix $\tilde{U}$, obtained from U by deleting the end points ± 1, there exists a function $f(x)$, continuous on $[-1, 1]$, such that

$$\lim_{n\to\infty} [f(x) - A_n(\tilde{U}; x)]^2 \, dx = \infty. \tag{28}$$

This example serves as an illustration of the general principle, which says that the laws of the asymptotic distribution of nodes are not fine enough to characterize completely the behavior of the alternating process.

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